

Support convergence for the spectrum of Wishart matrices with correlated entries

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1 Introduction

We consider Wishart matrices given by

$$Z_{n,m} = \frac{1}{n} \sum_{1 \leq p \leq m} X_p X_p^* \quad (1)$$

where $X_p, p \geq 0$ are i.i.d n -dimensional vectors. These matrices were intensively studied in connection with statistics. When the entries of X_p are independent, equidistributed and with finite second moment, it was shown by Marchenko and Pastur [17] that the empirical measure $L_n = n^{-1} \sum_{i=1}^n \delta_{\lambda_i}$ of the eigenvalues of such matrices converges almost surely as n, m go to infinity so that m/n goes to $c \in (0, +\infty)$. This result was extended by A.Pajor and L.Pastur [19] in the case where the entries of the X_p 's are correlated but have a log-concave isotropic law. Very recently, these authors together with O. Guédon and A. Lytova, proved the central limit theorem for the centered linear statistics [9] in the setting of [19]. To that end, they additionally assume that the law of the entries are “very good”, see [9, Definition 1.6], in the sense that mixed moments of degree four satisfy asymptotic conditions and quadratic forms satisfy concentration of measure property.

In this article we will consider also the case where the entries of the X_p 's are correlated but have a strictly log-concave law. We will show, under some symmetry and convergence hypotheses, that the central limit theorem for linear statistics holds around their limit, and deduce the convergence of the eigenvalues to the support of the limiting measure. To prove this result we shall assume that the law of the entries is “very good” in the sense of [9], but in fact even more that it is symmetric and with strictly log-concave law. The two later assumptions could possibly be removed.

The fluctuations of the spectral measure around the limiting measure or around the expectation were first studied by Jonsson [16] then by Pastur *et al.* in [13], and Sinai and Soshnikov [21] with $p \ll N^{1/2}$ possibly going to infinity with N . Since then, a long list of further-reaching results have been obtained: the central limit theorem was extended to the so-called matrix models where the entries interact via a potential in [15], the set of test functions was extended and the assumptions on the entries of the Wigner matrices weakened in [6, 2, 18, 20], Chatterjee developed a general approach to these questions in [7], under the condition that the law μ can be

written as a transport of the Gaussian law... Here, we will follow mainly the approach developed by Bai and Silverstein in [4] to study the fluctuations of linear statistics in the case where the vectors may have dependent entries. Since the fluctuations of the centered linear statistics were already studied in [9], we shall concentrate on the convergence of the mean of linear statistics (even though Bai and Silverstein method extends to obtain this result as well). We show that the mean, as the covariance (see [9]) will depend on several fourth joint moments of the entries of this vector.

Convergence of the support of the eigenvalues towards the support of the limiting measure was shown in [5] and [3] in the case of independent entries. We show that this convergence still holds in the case of dependent entries with log-concave distribution and therefore that such a dependency can not result in outliers.

1.1 Statement of the results

We consider a random matrix $Z_{n,m}$ given by (1), with m independent copies of a n -dimensional vector X whose entries maybe correlated. More precisely we assume that X follows the following distribution:

$$d\mathbb{P}(X) = \frac{1}{Z_n} e^{-V(X_1, \dots, X_n)} \prod_{i=1}^n dX_i.$$

Hypothesis 1. *We will assume that the mean of X is zero, that its covariance matrix is the identity matrix and that the four moments are homogeneous :*

$$\forall i \in \llbracket 1, n \rrbracket, \mathbb{E}[X_i^4] = \mu.$$

Besides, the following condition holds for the Hessian matrix of V :

$$\text{Hess}(V) \geq \frac{1}{C} I_n, \quad C > 0. \quad (2)$$

Moreover, the law of X is symmetric, that is $\text{Law}(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = \text{Law}(X_1, \dots, X_n)$ for all permutation σ of $\{1, \dots, n\}$.

This implies that concentration inequalities hold for the vector X (see the Appendix). In particular, we have $\text{Var}(\sum X_i^2) = O(n)$. Thus, it is natural to assume the following condition :

Hypothesis 2. *There exists $\kappa > 0$ so that*

$$\lim_{n \rightarrow \infty} n^{-1} \text{Var}(\sum X_i^2) = \kappa.$$

An example of potential V fulfilling both hypotheses 1 and 2 is given in section 4.3.

Let us consider the matrix $Y_{n,m}$ whose columns are m independent copies of the vector $\frac{1}{\sqrt{n}}X$. We denote by $W_{n,m}$ the symmetric block matrix of size $n + m$:

$$W_{n,m} = \begin{pmatrix} 0 & Y_{n,m}^* \\ Y_{n,m} & 0 \end{pmatrix}.$$

Let $F_{n,m}$ be the empirical spectral distribution of $W_{n,m}$ defined by :

$$F_{n,m}(dx) = \frac{1}{n+m} \sum_{i=1}^{n+m} \delta_{\lambda_i}.$$

Note that

$$\frac{1}{n} \text{Tr}(f(Z_{n,m})) = \frac{n+m}{2n} \int f(x^2) F_{n,m}(dx) + \frac{m-n}{2n} f(0)$$

so that the study of the eigenvalues of $W_{n,m}$ or $Z_{n,m}$ are equivalent. We will assume that

Hypothesis 3. $(m/n) \rightarrow c \in [1, \infty)$. Moreover, $cn - m$ converges towards σ as n, m go to infinity.

It has been proved by A. Pajor and L. Pastur that, when $n \rightarrow \infty, m \rightarrow \infty$ and $(m/n) \rightarrow c \in [1, \infty)$, the spectral measure L_n of $Z_{n,m}$ converges almost surely to the Marchenko-Pastur law. It is then not difficult to prove that $F_{n,m}$ converges almost surely to a probability measure F . In this paper, we will study the weak convergence of the centered measure $M_{n,m}(dx) = (n+m)(\mathbb{E}[F_{n+m}](dx) - F(dx))$.

Let A_K be the set of the set of C^K functions f . We will focus on the empirical process $M_{n,m} := M_{n,m}(f)$ indexed by A_K :

$$M_{n,m}(f) := (n+m) \int_{\mathbb{R}} f(x)(\mathbb{E}[F_{n,m}] - F)(dx), \quad f \in A_K.$$

Theorem 4. Under hypotheses 1, 2, and 3, there exists $K < \infty$ so that the process $M_{n,m} := (M_{n,m}(f))$ indexed by the set A_K converges to a process $M := \{M(f), f \in A_K\}$. Moreover, this process depends on both κ and μ .

As a non trivial corollary, we show that the support of the matrix $W_{n,m}$ converges towards the support of the law F , namely our main theorem:

Theorem 5. Under hypotheses 1, 2, and 3, the support of the eigenvalues of $W_{n,m}$ converges almost surely towards $[-1 - \sqrt{c}, 1 - \sqrt{c}] \cup [\sqrt{c} - 1, \sqrt{c} + 1]$.

Theorem 4 can be coupled with the central limit theorem for the centered statistics derived in [9] to derive the convergence of the random process

$$G_{n,m}(f) := (n+m) \int_{\mathbb{R}} f(x)(F_{n,m} - F)(dx), \quad f \in A_K.$$

The covariance of the limiting Gaussian process also depends on κ and μ , see [9].

1.2 Strategy of the proof

To prove Theorems 5 and 4, we shall prove that Theorem 4 holds when f is taken in the set $(z - x)^{-1}, z \in \mathbb{C}$. We then use a now standard strategy to generalize it to smooth enough functions that we describe below. More precisely, we denote by s_H the Stieltjes transform for the measure H defined by :

$$s_H(z) = \int_{\mathbb{R}} \frac{dH(x)}{x - z}, \quad z \notin \text{supp}(H),$$

we let $s_{n,m}(z)$ and $s(z)$ be the Stieltjes transform of $F_{n,m}$ and F respectively. We set $M_{n,m}$ to be the process

$$M_{n,m}(z) = (n+m)(\mathbb{E}[s_{n,m}(z)] - s)(z)$$

indexed by $z, \text{Im}z \neq 0$. We denote by \mathbb{C}_{v_0} the set $\mathbb{C}_{v_0} = \{z = u + iv, |v| \geq v_0, |z| < R\}$, where R is an arbitrary big enough constant. Then we shall prove:

Theorem 6. Assume Hypotheses 1, 2 and 3 hold. Then, there exists a constant $c > 0$ such that for $v_0 \geq n^{-c}$ the process $\{M_{n,m}(z); z \in \mathbb{C}_{v_0}\}$ converges uniformly to the process $\{M(z); z \in \mathbb{C}_{v_0}\}$

$$\begin{aligned} M(z) := & \sigma s^2(z) + (1 + c\partial_z s^2(z))(s^1(z))^3 \left[-\frac{\partial_z s^1(z)}{(s^1(z))^2} + c \left[(4\mu - 5)(s^2(z))^2 + 2\partial_z s^2(z) \right] \right] \\ & + (1 + \partial_z s^1(z))c(s^2(z))^3 \left[-\frac{\partial_z s^2(z)}{(s^2(z))^2} + \left[2(\mu - 2 + \frac{\kappa}{2})(s^1(z))^2 + 2\partial_z s^1(z) \right] \right] \end{aligned}$$

where

$$\begin{aligned} s^1(z) &= -\frac{1}{2z} \left(z^2 - c + 1 - \sqrt{(z^2 - c + 1)^2 - 4z^2} \right) \\ s^2(z) &= -\frac{1}{2cz} \left(z^2 + c - 1 - \sqrt{(z^2 - 1 + c)^2 - 4cz^2} \right). \end{aligned}$$

To deduce Theorems 5 and 4 from the above result, we rely on the following expression of [1, formula (5.5.11)], which allows to reconstruct integrals with respect to a measure from its Stieltjes transform. Namely, if f is a C^K compactly supported function, and if we set

$$\Psi_f(x, y) = \sum_{l=0}^K \frac{i^l}{l!} f^{(l)}(x) g(y) y^l$$

with $g : \mathbb{R} \mapsto [0, 1]$, g a smooth function with compact support, $g = 1$ near the origin and 0 outside $[-c_0, c_0]$, with c_0 an arbitrary constant, then for any probability measure μ on the real line

$$\int f(t) d\mu(t) = \Re \int_0^\infty dy \int_{-\infty}^\infty dx \left(\int \frac{\bar{\partial} \Psi(x, y)}{t - x - iy} \mu(dt) \right).$$

Here we have denoted $\bar{\partial} = \pi^{-1}(\partial_x + i\partial_y)$. Hence, we have

$$\begin{aligned} M_{n,m}(f) &= \mathbb{E} \left[\Re \int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \int \frac{\bar{\partial} \Psi_f(x, y)}{t - x - iy} (n + m) [F_{n,m} - F](t) dt \right] \\ &= \Re \int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \bar{\partial} \Psi_f(x, y) M_{n,m}(x + iy) \end{aligned}$$

Theorem 6 implies the convergence of the above integral for $y \geq n^{-c}$, with $M_{n,m}$ replaced by its limit M [note here that Ψ_f is compactly supported]. On the other hand, $|\bar{\partial} \Psi_f(x, y)|/y^K$ and $n^{-1}y M_{n,m}(x + iy)$ are uniformly bounded, so that if $Kc > 1$, the integral over $[0, n^{-c}]$ is neglectable. Hence, we conclude that for such functions f , we have

$$\lim_{n \rightarrow \infty} M_{n,m}(f) = \Re \int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \bar{\partial} \Psi_f(x, y) M(x + iy). \quad (3)$$

This convergence extends to non-compactly supported C^K functions by the rough estimates on the eigenvalues derived in Lemma 15. This completes the proof of Theorem 4.

To deduce from (3) the convergence of the support of the empirical measure, that is Theorem 5, we finally take f C^K and vanishing on the support of the Pastur-Marchenko distribution. But then Ψ_f also vanishes when x belongs to the support of the Pastur-Marchenko distribution. Since $M(x + iy)$ is analytic away from this support (as it is a smooth function of s^1 and s^2 which are analytic there), $\bar{\partial} M(x + iy)$ vanishes on the support of integration. This shows after an integration by parts that

$$\begin{aligned} \lim_{n \rightarrow \infty} M_{n,m}(f) &= \Re \int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \bar{\partial} [\Psi_f(x, y) M(x + iy)] \\ &= \Re [i\pi^{-1} \int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \partial_y [\Psi_f(x, y) M(x + iy)]] \end{aligned}$$

where we noticed that the integral of $\partial_x [\Psi_f(x, y) M(x + iy)]$ vanishes as $\Psi_f(x, y)$ vanishes when x is outside a compact. On the other hand

$$\int_0^{+\infty} dy \int_{-\infty}^{+\infty} dx \partial_y [\Psi_f(x, y) M(x + iy)] = i\pi^{-1} \int f(x) M(x) dx$$

is purely imaginary. Hence, we conclude that for any f compactly supported, C^K and vanishing on the support of the Pastur-Marchenko distribution, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}[\sum f(\lambda_i)] = \lim_{n \rightarrow \infty} M_{n,m}(f) = 0.$$

Taking f non-negative and greater than one on some compact which does not intersect the support of the Pastur-Marchenko law shows that the probability that there are eigenvalues in this compact as n goes to infinity vanishes. By Lemma 15, we conclude that the probability that there is an eigenvalue at distance ϵ of the support of the Pastur-Marchenko law goes to zero as n goes to infinity. But we also have concentration of the extreme eigenvalues Theorem 14 and therefore the convergence holds almost surely.

2 A few useful results

In this section, we review a few classical results about the convergence of $s_{n,m}$ and provide some proofs on which we shall elaborate to derive Theorem 6. In particular, we emphasis on which domain the convergence holds, in preparation to the proof of Theorem 5.

To simplify the computations, we are going to introduce a few notations. Let

$$S = (W_{n,m} - z)^{-1}.$$

We write α_k the vector obtained from the k -th column $W_{n,m}$ by deleting the k -th entry and $W_{n,m}(k)$ the matrix resulting from deleting the k -th row and column from $W_{n,m}$. Let S_k be $S_k = (W_{n,m}(k) - z)^{-1}$, we write S_1 (respectively S_2) the submatrix of order n formed by the last n row and columns (respectively the submatrix of order m formed by the first m row and columns). Here are some useful notations :

$$\begin{aligned} s_{n,m}^1(z) &= \frac{1}{n} \text{Tr}(S_1) & s_{n,m}^2(z) &= \frac{1}{m} \text{Tr}(S_2) \\ s_{n,m}(z) &= \frac{1}{n+m} \text{Tr}(S) = \frac{ns_{n,m}^1(z) + ms_{n,m}^2(z)}{n+m}. \\ \beta_k &= z + \frac{1}{n} \alpha_k^* S_k \alpha_k, \\ \epsilon_k &= -\frac{1}{n} \alpha_k^* S_k \alpha_k + \mathbb{E}[s_{n,m}^1(z)], & k \leq m \\ \epsilon_k &= -\frac{1}{n} \alpha_k^* S_k \alpha_k + c_{n,m} \mathbb{E}[s_{n,m}^2(z)], & k > m, \quad c_{n,m} = \frac{m}{n} \\ \delta_1(z) &= -\frac{1}{n} \sum_{k=m+1}^{n+m} \frac{\epsilon_k}{\beta_k(z + c_{n,m} \mathbb{E}[s_{n,m}^2(z)])}, \\ \delta_2(z) &= -\frac{1}{m} \sum_{k=1}^m \frac{\epsilon_k}{\beta_k(z + \mathbb{E}[s_{n,m}^1(z)])}, \end{aligned} \tag{4}$$

Remark 7. Since $W_{n,m}$ is a real symmetric random matrix, the eigenvalues of the matrix S can be written as $\frac{1}{\lambda_j - z}$, with λ_j the real eigenvalues of the matrix $W_{n,m}$. Thus, all the eigenvalues are bounded by $\frac{1}{|\Im z|}$. We also have $|\text{Tr}(S)| \leq \frac{(n+m)}{|\Im z|}$ and $|\text{Tr}(S^2)| \leq \frac{(n+m)}{|\Im z|^2}$. Moreover, we have $\frac{dS(z)}{dz} = S^2(z)$.

Theorem 8. For $v_0 \geq n^{-\frac{1}{13}}$, uniformly on $z \in \mathbb{C}_{v_0}$, $s_{n,m}(z)$ goes to $s(z)$ in L_2 , where

$$s(z) = -\frac{1}{1+c} \left(z - \frac{\sqrt{(z^2 - c + 1)^2 - 4z^2}}{z} \right) = \frac{s^1(z) + cs^2(z)}{1+c}$$

with for $i = 1, 2$

$$s^i(z) = -\frac{1}{2c^{i-1}z} \left(z^2 + (-1)^i(c-1) - \sqrt{(z^2 + (-1)^i(c-1))^2 - 4c^{i-1}z^2} \right).$$

Proof. By Lemma 17 in the appendix, we have

$$\text{Tr}(S_1) = -\sum_{k=m+1}^{n+m} \frac{1}{\beta_k}$$

where, for $k > m$, we have denoted

$$\frac{1}{\beta_k} = \frac{1}{z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)] - \epsilon_k} = \frac{1}{z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)]} + \frac{\epsilon_k}{(z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)] - \epsilon_k)(z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)])}.$$

Thus,

$$s_{n,m}^1(z) = -\frac{1}{z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)]} + \delta_1(z).$$

By the same method, we obtain

$$s_{n,m}^2(z) = -\frac{1}{z + \mathbb{E}[s_{n,m}^1(z)]} + \delta_2(z).$$

We can deduce, by taking the mean in both previous equalities, that $\mathbb{E}[s_{n,m}^1(z)]$ and $\mathbb{E}[s_{n,m}^2(z)]$ are solutions of the following system

$$\begin{cases} \mathbb{E}[s_{n,m}^1(z)] = -\frac{1}{z + c_{n,m}\mathbb{E}[s_{n,m}^2(z)]} + \mathbb{E}[\delta_1(z)] \\ \mathbb{E}[s_{n,m}^2(z)] = -\frac{1}{z + \mathbb{E}[s_{n,m}^1(z)]} + \mathbb{E}[\delta_2(z)] \end{cases}.$$

Thus, $\mathbb{E}[s_{n,m}^1(z)]$ is a solution of the quadratic equation

$$a_2\mathbb{E}[s_{n,m}^1(z)]^2 + a_1\mathbb{E}[s_{n,m}^1(z)] + a_0 = 0$$

with

$$\begin{aligned} a_2 &= z + c_{n,m}\mathbb{E}[\delta_2(z)] \\ a_1 &= z^2 - c_{n,m} + 1 + zc_{n,m}\mathbb{E}[\delta_2(z)] - z\mathbb{E}[\delta_1(z)] - c_{n,m}\mathbb{E}[\delta_1(z)]\mathbb{E}[\delta_2(z)] \\ a_0 &= z - z^2\mathbb{E}[\delta_1(z)] + c_{n,m}\mathbb{E}[\delta_1(z)] - zc_{n,m}\mathbb{E}[\delta_1(z)]\mathbb{E}[\delta_2(z)]. \end{aligned} \tag{5}$$

We deduce that

$$\mathbb{E}[s_{n,m}^1(z)] = -\frac{\left(a_1 - \sqrt{a_1^2 - 4a_0a_2}\right)}{2a_2}. \tag{6}$$

We only need to estimate a_0 , a_1 and a_2 to find the limit of $\mathbb{E}[s_{n,m}^1(z)]$. We now show that $\mathbb{E}[\delta_1(z)] = O(\frac{1}{n})$ and $\mathbb{E}[\delta_2(z)] = O(\frac{1}{n})$. Using the fact that, for $k \leq m$,

$$\frac{1}{\beta_k} = \frac{1}{z + \mathbb{E}[s_{n,m}^1(z)] - \epsilon_k} = \frac{1}{z + \mathbb{E}[s_{n,m}^1(z)]} + \frac{\epsilon_k}{(z + \mathbb{E}[s_{n,m}^1(z)] - \epsilon_k)(z + \mathbb{E}[s_{n,m}^1(z)])},$$

we can write

$$\mathbb{E}[\delta_2(z)] = -\frac{1}{m} \sum_{k=1}^m \mathbb{E} \left[\frac{\epsilon_k}{(z + \mathbb{E}[s_{n,m}^1(z)])^2} + \frac{\epsilon_k^2}{\beta_k(z + \mathbb{E}[s_{n,m}^1(z)])^2} \right].$$

On the other hand, we have $|\beta_k| \geq |\Im(\beta_k)| = |\Im(z)| \geq v_0$ and $|z + \mathbb{E}[s_{n,m}^1(z)]| \geq v_0$, which yields

$$|\mathbb{E}[\delta_2(z)]| \leq \frac{v_0^{-2} + v_0^{-3}}{m} \sum_{k=1}^m (|\mathbb{E}[\epsilon_k]| + \mathbb{E}[\epsilon_k^2]). \quad (7)$$

To estimate the first term note that we have

$$\mathbb{E}[\epsilon_k] = \frac{1}{n} \text{Tr}(S_1 - (S_k)_1). \quad (8)$$

To compute this term, we will use the following equality :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BF^{-1}CA^{-1} & -A^{-1}BF^{-1} \\ -F^{-1}CA^{-1} & F^{-1} \end{pmatrix},$$

where $F = D - CA^{-1}B$. Since S_1 is the $n \times n$ block in the right bottom, if we let A, B and C be $A = D = -zI_n$, $B = Y_{n,m}^*$ and $C = Y_{n,m}$, we have $S_1 = F^{-1} = -zI_n + z^{-1}Y_{n,m}Y_{n,m}^*$. Therefore, we have

$$\text{Tr}(S_1) = \text{Tr}(z^{-1}Y_{n,m}Y_{n,m}^* - z)^{-1}.$$

Using the notation $Y_{n,m}(k)$ to denote the submatrix of $Y_{n,m}$ obtained by deleting its k -th column (its size is $n \times (n-1)$), we find by a similar method

$$\text{Tr}(S_k)_1 = \text{Tr}(z^{-1}Y_{n,m}(k)Y_{n,m}(k)^* - z)^{-1}.$$

Therefore, we deduce that

$$\text{Tr}(S_1 - (S_k)_1) = z \left[\text{Tr}(Y_{n,m}Y_{n,m}^* - z^2)^{-1} - \text{Tr}(Y_{n,m}(k)Y_{n,m}(k)^* - z^2)^{-1} \right].$$

But, $Y_{n,m}(k)Y_{n,m}(k)^* = Y_{n,m}Y_{n,m}^* - \frac{1}{n}\alpha_k\alpha_k^*$, so that we have proved the equality

$$\text{Tr}(S_1 - (S_k)_1) = -z \frac{1}{n} \alpha_k^* (Y_{n,m}Y_{n,m}^* - z^2)^{-1} (Y_{n,m}(k)Y_{n,m}(k)^* - z^2)^{-1} \alpha_k. \quad (9)$$

Since $Y_{n,m}Y_{n,m}^*$ is a non-negative hermitian matrix, its eigenvalues are non-negative. The eigenvalues of $(Y_{n,m}Y_{n,m}^* - z^2)^{-1}$ can be written as $1/(\lambda - z^2)$ or, as we prefer, $1/(\sqrt{\lambda} - z)(\sqrt{\lambda} + z)$. Thus, the absolute value of the eigenvalues of $(Y_{n,m}Y_{n,m}^* - z^2)^{-1}$ are all bounded by v_0^{-2} . The same reasoning holds for $(Y_{n,m}(k)Y_{n,m}(k)^* - z^2)^{-1}$. Therefore, we deduce from (9) that

$$|\mathbb{E}[\text{Tr}(S_1 - (S_k)_1)]| \leq |z| v_0^{-4} \frac{1}{n} \mathbb{E}[\|\alpha_k\|^2].$$

But, by Hypothesis 1, we have

$$\mathbb{E}[\|\alpha_k\|^2] = \mathbb{E}\left[\sum_{i=1}^n X_i^2\right] = n.$$

Thus, we have shown according to (8) that

$$|\mathbb{E}[\epsilon_k]| \leq \frac{1}{n} R v_0^{-4}$$

Moreover,

$$\mathbb{E}[\epsilon_k^2] \leq 2 \left(\mathbb{E} \left[\frac{1}{n^2} |\alpha_k^* S_k \alpha_k - \text{Tr}(S_k)_1|^2 \right] + \mathbb{E} \left[\frac{1}{n^2} |\text{Tr}((S_k)_1 - S_1)|^2 \right] \right).$$

The concentration of measure Theorem 11 states that the first term is of order $O(n^{-1})$. For the second one, by the previous computation, we deduce that

$$|\text{Tr}((S_k)_1 - S_1)|^2 \leq R^2 v_0^{-8} \frac{1}{n^2} \|\alpha_k\|^4. \quad (10)$$

Since $\mathbb{E}[\frac{1}{n^2} \|\alpha_k\|^4] = O(1)$, we conclude that $\mathbb{E}[\epsilon_k^2] = O(v_0^{-8} n^{-1})$.

We have now shown according to (7) that

$$\mathbb{E}[\delta_2(z)] = O\left(\frac{v_0^{-12}}{n}\right). \quad (11)$$

We find a similar bound for $\mathbb{E}[\delta_1(z)]$. Since $\mathbb{E}[\delta_i(z)] = O(v_0^{-12} n^{-1})$, $i = 1, 2$, a Taylor expansion gives us

$$\begin{aligned} \mathbb{E}[s_{n,m}^1(z)] &= -\frac{1}{2z} \left(z^2 - c + 1 - \sqrt{(z^2 - c + 1)^2 - 4z^2} \right) + O\left(\frac{v_0^{-12}}{n}\right). \\ \mathbb{E}[s_{n,m}^2(z)] &= -\frac{1}{2cz} \left(z^2 + c - 1 - \sqrt{(z^2 - 1 + c)^2 - 4cz^2} \right) + O\left(\frac{v_0^{-12}}{n}\right). \end{aligned} \quad (12)$$

Since $\mathbb{E}[s_{n,m}(z)] = \frac{n\mathbb{E}[s_{n,m}^1(z)] + m\mathbb{E}[s_{n,m}^2(z)]}{n+m}$, we conclude that

$$\mathbb{E}[s_{n,m}(z)] = \frac{s^1(z) + cs^2(z)}{1+c} + O\left(\frac{v_0^{-12}}{n}\right).$$

Hence $\mathbb{E}[s_{n,m}(z)]$ converges towards $s(z)$, uniformly on \mathbb{C}_{v_0} for $v_0 \geq n^{-\frac{1}{13}}$.

Now, for the convergence of $s_{n,m}(z)$ to $s(z)$ in L_2 , let f be the function $f : x \mapsto (x - z)^{-1}$. f is a Lipschitz function, whose Lipschitz constant is bounded by v_0^{-2} . Corollary 13 states that the variance of $\text{Tr}(S)$ is uniformly bounded for all (n, m) . From that, we see that

$$\mathbb{E}[|s_{n,m}(z) - \mathbb{E}[s_{n,m}(z)]|^2] = O(v_0^{-4} n^{-2}).$$

Thus, $s_{n,m}(z)$ converges to $s(z)$ in L_2 .

□

Theorem 9. For $v_0 \geq n^{-\frac{1}{16}}$, uniformly on $z \in \mathbb{C}_{v_0}$,

$$\mathbb{E}[\max_{i \leq m} |S_{ii} - s^2(z)|^2] = O(v_0^{-16} n^{-1}) \quad \text{and} \quad \mathbb{E}[\max_{i > m} |S_{ii} - s^1(z)|^2] = O(v_0^{-16} n^{-1}).$$

Proof. Let $i \leq m$. By definition, we have

$$\begin{aligned} S_{ii} &= -\frac{1}{z + \frac{1}{n} \alpha_i^* S_i \alpha_i} \\ &= \frac{1}{-z - s^1(z)} + \frac{-s^1(z) + \frac{1}{n} \alpha_i^* S_i \alpha_i}{(-z - \frac{1}{n} \alpha_i^* S_i \alpha_i)(-z - s^1(z))}. \end{aligned}$$

But, we know that $s^2(z) = (-z - s^1(z))^{-1}$. We thus deduce that

$$S_{ii} - s^2(z) = \frac{-s^1(z) + \frac{1}{n} \alpha_i^* S_i \alpha_i}{(-z - \frac{1}{n} \alpha_i^* S_i \alpha_i)(-z - s^1(z))}.$$

Since $z \in \mathbb{C}_{v_0}$, there exist $K > 0$ such that $|(-z - \frac{1}{n}\alpha_i^* S_i \alpha_i)(-z - s^1(z))| > K v_0^2$. Therefore, we have

$$\begin{aligned} |S_{ii} - s^2(z)| &\leq K v_0^{-2} | -s^1(z) + \frac{1}{n} \alpha_i^* S_i \alpha_i | \\ &\leq K v_0^{-2} \left(\frac{1}{n} |\alpha_i^* S_i \alpha_i - \mathbb{E}[\text{Tr}(S_i)_1]| + \frac{1}{n} |\mathbb{E}[\text{Tr}((S_i)_1 - S_1)]| + |\mathbb{E}[s_{n,m}^1(z)] - s^1(z)| \right). \end{aligned}$$

Both last terms converge uniformly to 0 for all i in L_2 (this result has been proved previously) in $O(v_0^{-12} n^{-1})$. For the first one, using the concentration inequality, we can show that L^2 norm of the first term is in $O(v_0^{-2} n^{-1})$, where the $O(v_0^{-2} n^{-1})$ is uniform for all i (indeed, the constants are independent of i), which concludes the proof for $i \leq m$. The proof is similar for $i > m$.

Therefore, for $i > m$, $\mathbb{E}[|S_{ii} - s^1(z)|^2] = O(v_0^{-16} n^{-1})$, where the upper bound is uniform for all i . □

3 Convergence of the process $M_{n,m}(z)$

We are going to compute the limit of the function of $M_{n,m}(z)$.

Theorem 10. *Under the hypotheses 1, 2 and 3, uniformly on $z \in \mathbb{C}_{v_0}$, $v_0 \geq n^{-1/20}$, the function $M_{n,m}(z)$ converges to the function M defined by*

$$\begin{aligned} M(z) := & \sigma s^2(z) + (1 + c \partial_z s^2(z))(s^1(z))^3 \left[-\frac{\partial_z s^1(z)}{(s^1(z))^2} + c \left[(4\mu - 5)(s^2(z))^2 + 2\partial_z s^2(z) \right] \right] \\ & + (1 + \partial_z s^1(z))c(s^2(z))^3 \left[-\frac{\partial_z s^2(z)}{(s^2(z))^2} + \left[2(\mu - 2 + \frac{\kappa}{2})(s^1(z))^2 + 2\partial_z s^1(z) \right] \right] \end{aligned}$$

where

$$\begin{aligned} s^1(z) &= -\frac{1}{2z} \left(z^2 - c + 1 - \sqrt{(z^2 - c + 1)^2 - 4z^2} \right) \\ s^2(z) &= -\frac{1}{2cz} \left(z^2 + c - 1 - \sqrt{(z^2 - 1 + c)^2 - 4cz^2} \right) \end{aligned}$$

Proof. Recall that $M_{n,m}(z) = (n + m)(\mathbb{E}[s_{n,m}(z)] - s(z))$ can be written

$$M_{n,m}(z) = n(\mathbb{E}[s_{n,m}^1(z)] - s^1(z)) + m(\mathbb{E}[s_{n,m}^2(z)] - s^2(z)) + o(1).$$

Recall that we have from (5) that

$$\mathbb{E}[s_{n,m}^1(z)] = -\frac{\left(a_1 - \sqrt{a_1^2 - 4a_0 a_2} \right)}{2a_2}$$

with

$$\begin{aligned} a_2 &= z + c\mathbb{E}[\delta_2(z)] + O(v_0^{-24} n^{-2}) \\ a_1 &= z^2 - c + 1 + (c - c_{n,m}) + zc\mathbb{E}[\delta_2(z)] - z\mathbb{E}[\delta_1(z)] + O(v_0^{-24} n^{-2}) \\ a_0 &= z - z^2\mathbb{E}[\delta_1(z)] + c\mathbb{E}[\delta_1(z)] + O(v_0^{-24} n^{-2}) \end{aligned}$$

By a Taylor expansion, we deduce

$$\begin{aligned} \sqrt{\left(\frac{a_1}{a_2}\right)^2 - 4\frac{a_0}{a_2}} &= \sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4} \\ &+ \frac{1}{\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[\frac{(z^2 - c + 1)}{z} \frac{1}{z^2} \left(z(c - c_{n,m}) - z^2 \mathbb{E}[\delta_1(z)] + c(c - 1) \mathbb{E}[\delta_2(z)] \right) \right] \\ &- \frac{1}{\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[\frac{2c}{z} (\mathbb{E}[\delta_1(z)] - \mathbb{E}[\delta_2(z)]) - 2z \mathbb{E}[\delta_1(z)] \right] + O(v_0^{-25} n^{-2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[s_{n,m}^1(z) - s^1(z)] &= -\frac{1}{2z^2} \left(z(c - c_{n,m}) - z^2 \mathbb{E}[\delta_1(z)] + c(c - 1) \mathbb{E}[\delta_2(z)] \right) \\ &+ \frac{1}{2\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[\frac{(z^2 - c + 1)}{z} \frac{1}{z^2} \left(z(c - c_{n,m}) - z^2 \mathbb{E}[\delta_1(z)] + c(c - 1) \mathbb{E}[\delta_2(z)] \right) \right] \\ &- \frac{1}{2\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[\frac{2c}{z} (\mathbb{E}[\delta_1(z)] - \mathbb{E}[\delta_2(z)]) - 2z \mathbb{E}[\delta_1(z)] \right] + O(v_0^{-26} n^{-2}). \end{aligned}$$

For $\mathbb{E}[s_{n,m}^2(z)]$, we have similarly

$$\begin{aligned} \mathbb{E}[s_{n,m}^2(z) - s^2(z)] &= -\frac{1}{2c^2 z^2} \left(-c^2 z^2 \mathbb{E}[\delta_2(z)] + (c_{n,m} - c)(z - z^3) - c(c - 1) \mathbb{E}[\delta_1(z)] \right) \\ &+ \frac{1}{2\sqrt{\frac{(z^2 + c - 1)^2}{(cz)^2} - \frac{4}{c}}} \left[\frac{(z^2 + c - 1)}{cz} \frac{1}{-c^2 z^2} \left(-c^2 z^2 \mathbb{E}[\delta_2(z)] + (c_{n,m} - c)(z - z^3) - c(c - 1) \mathbb{E}[\delta_1(z)] \right) \right] \\ &- \frac{1}{2\sqrt{\frac{(z^2 + c - 1)^2}{(cz)^2} - \frac{4}{c}}} \left[\frac{2}{c^2 z} \left(c(1 - z^2) \mathbb{E}[\delta_2(z)] - (c_{n,m} - c)z - c \mathbb{E}[\delta_1(z)] \right) \right] + O(v_0^{-26} n^{-2}). \end{aligned}$$

To find the limit of $M_{n,m}(z)$, we only need to find an equivalent of $c - c_{n,m}$, of $\mathbb{E}[\delta_1(z)]$ and of $\mathbb{E}[\delta_2(z)]$. First, we have by Hypothesis 3, $(c - c_{n,m}) = \frac{nc-m}{n} \sim \frac{\sigma}{n}$. To compute an equivalent of $\mathbb{E}[\delta_2(z)]$, we will use the following formula

$$\frac{1}{u - \epsilon} = \frac{1}{u} + \frac{\epsilon}{u^2} + \frac{\epsilon^2}{u^2(u - \epsilon)}.$$

Applied to β_k , it gives

$$\frac{1}{\beta_k} = \frac{1}{z + \mathbb{E}[s_{n,m}^1(z)] - \epsilon_k} = \frac{1}{z + \mathbb{E}[s_{n,m}^1(z)]} + \frac{\epsilon_k}{(z + \mathbb{E}[s_{n,m}^1(z)])^2} + \frac{\epsilon_k^2}{\beta_k(z + \mathbb{E}[s_{n,m}^1(z)])^2}.$$

Finally, we can write

$$\begin{aligned} \delta_2(z) &= -\frac{1}{m} \sum_{k=1}^m \frac{\epsilon_k}{(z + \mathbb{E}[s_{n,m}^1(z)])^2} - \frac{1}{m} \sum_{k=1}^m \frac{\epsilon_k^2}{(z + \mathbb{E}[s_{n,m}^1(z)])^3} - \frac{1}{m} \sum_{k=1}^m \frac{\epsilon_k^3}{\beta_k(z + \mathbb{E}[s_{n,m}^1(z)])^3} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

Let us find the limit of the expectation of each term.

First, since $z \in \mathbb{C}_{v_0}$, by the concentration Theorem 11 applied to $A = S_k(z)$ we find for all $p \in \mathbb{N}$ a finite constant C_p such that for all k

$$\mathbb{E}[|\epsilon_k|^p] \leq \frac{C_p}{(\sqrt{n}v_0)^p}. \quad (13)$$

This implies that

$$|\mathbb{E}[S_3]| \leq v_0^{-4} \frac{1}{m} \sum_{k=1}^m \mathbb{E}[|\epsilon_k|^3] = O\left(\frac{v_0^{-7}}{n\sqrt{n}}\right).$$

Then, for $k \leq m$, using a similar computation as done in the proof of Theorem 8, we have

$$\mathbb{E}[\epsilon_k] = \frac{1}{n} \mathbb{E}[\text{Tr}(S_1 - (S_k)_1)].$$

By [8, Lemma 3.2], we have

$$\text{Tr}(S_1 - (S_k)_1) = \sum_{i=1}^n \frac{(W_{n,m} - z)_{ki}^{-1} (W_{n,m} - z)_{ik}^{-1}}{(W_{n,m} - z)_{kk}^{-1}} = \frac{(W_{n,m} - z)_{kk}^{-2}}{(W_{n,m} - z)_{kk}^{-1}} = \frac{((W_{n,m} - z)_{kk}^{-1})'}{(W_{n,m} - z)_{kk}^{-1}}$$

Therefore, we deduce that

$$n\mathbb{E}[\epsilon_k] = \mathbb{E}[\text{Tr}(S_1 - (S_k)_1)] = \mathbb{E}\left[\frac{((W_{n,m} - z)_{kk}^{-1})'}{(W_{n,m} - z)_{kk}^{-1}}\right]$$

Theorem 9 states that $(W_{n,m} - z)_{kk}^{-1}$ goes to $s^2(z)$ in L_2 provided $v_0 \geq n^{-1/20}$. Moreover, we may assume without loss of generality that the eigenvalues of $W_{n,m}$ are bounded by some Λ as by Lemma 15 for Λ big enough

$$E[1_{\|W_{n,m}\| \geq \Lambda} \epsilon_k] \leq 2nv_0^{-1} e^{-\alpha\Lambda n}.$$

On $\|W_{n,m}\| \leq \Lambda$, $(W_{n,m} - z)_{kk}^{-1}$ is lower bounded by a constant $C(\Lambda) > 0$ and hence

$$|n\mathbb{E}[\epsilon_k] - \frac{1}{s^2(z)} \partial_z \mathbb{E}[(W_{n,m} - z)_{kk}^{-1}]| \leq 2nv_0^{-1} e^{-\alpha\Lambda n} + C(\Lambda)^{-2} v_0^{-2} n^{-1} v_0^{-16}.$$

Finally $z \rightarrow \mathbb{E}[(W_{n,m} - z)_{kk}^{-1}]$ is analytic, and bounded by Theorem 9 provided $v_0 \geq n^{-1/16}$. Hence, writing Cauchy formula, we check that its derivative converges towards the derivative of s^2 for $v_0 \geq n^{-1/19}$. Therefore, we conclude that uniformly on $v_0 \geq n^{-1/19}$,

$$n\mathbb{E}[\epsilon_k] - \frac{\partial_z s^2(z)}{s^2(z)} \rightarrow 0.$$

The last term to compute is $\mathbb{E}[\epsilon_k^2]$. We write

$$\mathbb{E}[\epsilon_k^2] = \text{Var}(\epsilon_k) + \mathbb{E}[\epsilon_k]^2 = \text{Var}(\epsilon_k) + O\left(\frac{1}{n^2}\right).$$

We only need to compute the limit of the variance of ϵ_k .

$$\text{Var}(\epsilon_k) = \frac{1}{n^2} \text{Var}(\alpha_k^* S_k \alpha_k).$$

We decompose this variance as follows

$$\text{Var}(\alpha_k^* S_k \alpha_k) = T_1^n + T_2^n + T_3^n + T_4^n + T_5^n + T_6^n \quad (14)$$

where if we denote in short $\alpha_k = (X_i)_i$ and $(S_k)_1 = (s_{ij})_{ij}$ we have

$$\begin{aligned}
T_1^n &= \sum_{i,j} \text{Var}(X_i s_{ij} X_j) \\
T_2^n &= \sum_{i,p} \text{Cov}(s_{ii} X_i^2, s_{pp} X_p^2) \\
T_3^n &= 2 \sum_i \sum_{p \neq q} \text{Cov}(X_i^2 s_{ii}, X_p s_{pq} X_q) \\
T_4^n &= 2 \sum_i \sum_{j \neq p} \text{Cov}(X_i s_{ij} X_j, X_p s_{pi} X_i) \\
T_5^n &= \sum_{i,j} \text{Cov}(X_i s_{ij} X_j, X_j s_{ji} X_i) \\
T_6^n &= \sum_{i \neq j \neq p \neq q} \text{Cov}(X_i s_{ij} X_j, X_p s_{pq} X_q)
\end{aligned}$$

We shall prove that $n^{-1}T_i^n$ converges for $i = 1, 2, 5$ to a non zero limit whereas for $i = 3, 4, 6$ it goes to zero. Let us compute an equivalent for each term. On the way, we shall use the symmetry of the law of X , which implies that the law of the matrix $W_{n,m}$ is invariant under $(W_{n,m}(ij)) \rightarrow (W_{n,m}(\sigma(i)\sigma(j)))$ for any permutation σ keeping fixed $\{1, \dots, m\}$ but permuting $\{m+1, \dots, n+m\}$. Therefore, we deduce for instance that $\mathbb{E}[s_{ii}s_{pq}] = \mathbb{E}[s_{11}s_{23}]$ for all distinct $i, p, q < m$. We shall now estimate these terms by taking advantage of some linear algebra tricks used already in e.g. [8, Lemma 3.2].

Let \mathbb{T} be a set of $\llbracket 1, (n+m) \rrbracket$. Let us denote by $W_{n,m}^{(\mathbb{T})}$ the submatrix of $W_{n,m}$ obtained by deleting the i -th row and column, for $i \in \mathbb{T}$. To simplify the notations, let us denote by $(ij\mathbb{T})$ the set $(i \cup j \cup \mathbb{T})$.

Let us introduce the following notations :

$$Z_{ij}^{(\mathbb{T})} = \frac{1}{n} \sum_{p,q} (\alpha_i)_p (W_{n,m}^{(\mathbb{T})} - z)^{-1}(p, q) (\alpha_j)_q,$$

$$K_{ij}^{(\mathbb{T})} = \frac{1}{\sqrt{n}} (X_j)_i - z \delta_{ij} - Z_{ij}^{(\mathbb{T})}.$$

We have the following formulas for $i \neq j \neq k$

$$s_{ij} = (W_{n,m}^{(k)} - z)^{-1}(i, j) = -s_{jj} s_{ii}^{(j)} K_{ij}^{(ijk)} \quad (15)$$

$$s_{ii} = s_{ii}^j + s_{ij} s_{ji} s_{jj}^{-1} \quad (16)$$

$$s_{ij} = s_{ij}^k + s_{ik} s_{kj} s_{kk}^{-1} \quad (17)$$

The concentration inequality states that, for $i \neq j$, $\mathbb{E}[|Z_{ij}^{(\mathbb{T})}|^p] = O(v_0^{-p} n^{-\frac{p}{2}})$. Thus,

$$\mathbb{E}[|K_{ij}^{(\mathbb{T})}|^p] = O(v_0^{-p} n^{-\frac{p}{2}}). \quad (18)$$

As a consequence, (15) implies that for all p , for $i \neq j$

$$\mathbb{E}[|s_{ij}|^p] = O(v_0^{-3p} n^{-\frac{p}{2}}). \quad (19)$$

Moreover, if $z = E + i\eta$, we find

$$\Im s_{jj}(z) = \langle e_j, \frac{\eta}{\eta^2 + (E - W)^2} e_j \rangle \geq \frac{\eta}{(2M)^2}$$

on $\|W\| \leq M$, $\eta \leq M$. Hence, we can use Lemma 15 to find that

$$\mathbb{E}[|s_{jj}^{-1}|^p] = O(\frac{1}{v_0^p})$$

Therefore, from (16) and (19), we deduce

$$\mathbb{E}[|s_{ii} - s_{ii}^{(j)}|^p] = O(v_0^{-3p} n^{-p}), \quad \mathbb{E}[|s_{ij} - s_{ij}^{(k)}|^p] = O(v_0^{-3p} n^{-p}). \quad (20)$$

We next show that for all $i \neq p \neq q$

$$\mathbb{E}[s_{ii} s_{pq}] = O(\frac{v_0^{-9}}{n\sqrt{n}}) \quad (21)$$

Let us first bound

$$\mathbb{E}[s_{pq}] = -\mathbb{E}[s_{pp} s_{qq}^{(p)} K_{pq}^{(pqk)}] = -\mathbb{E}[s_{pp}^{(q)} s_{qq}^{(p)} K_{pq}^{(pqk)}] + O(\frac{v_0^{-12}}{n\sqrt{n}})$$

by (20) and (23). Denoting $e_p := s_{pp}^q - s^1(z)$ and $e_q := s_{qq}^{(p)} - s^1(z)$, we have

$$\begin{aligned} \mathbb{E}[s_{pq}] &= \mathbb{E}[-s_{pp}^q s_{qq}^{(p)} K_{pq}^{(pqk)}] = -\mathbb{E}[(e_p + s^1(z))(e_q + s^1(z)) K_{pq}^{(pqk)}] \\ &= s^1(z) \mathbb{E}[e_p K_{pq}^{(pqk)}] + s^1(z) \mathbb{E}[e_q K_{pq}^{(pqk)}] + \mathbb{E}[e_p e_q K_{pq}^{(pqk)}]. \end{aligned}$$

where we used that $\mathbb{E}[K_{pq}^{(pqk)}] = 0$, because $p \neq q$. Moreover, recall that when we estimated e_p in the proof of Theorem 9, we had

$$e_p = \frac{-s^1(z) + \frac{1}{n} \alpha_p^* s^{(qp)} \alpha_p}{(z + s^1(z))^2} + O(\frac{1}{v_0^3} | -s^1(z) + \frac{1}{n} \alpha_p^* s \alpha_p |^2) = \frac{\frac{1}{n} \alpha_p^* s^{(qp)} \alpha_p - \frac{1}{n} \text{Tr} s^{(qp)}}{(z + s^1(z))^2} + O(v_0^{-19} n^{-1})$$

using the independence of α_p and α_q , and their centering, we see that for $r = p$ or q

$$\mathbb{E}[(\frac{1}{n} \alpha_p^* s^{(qp)} \alpha_p - \frac{1}{n} \text{Tr} s^{(qp)}) K_{pq}^{(pqk)}] = 0.$$

Hence we deduce that

$$\mathbb{E}[s_{pq}] = O(\frac{v_0^{-20}}{n\sqrt{n}}). \quad (22)$$

It remains to bound

$$\mathbb{E}[s_{pq}(s_{ii} - s^1(z))] = \mathbb{E}[s_{pp} s_{qq}^{(p)} K_{pq}^{(pqk)}(s_{ii} - s^1(z))].$$

Thanks to (20) and (19), we can replace s_{pp} (resp. $s_{qq}^{(p)}$, resp. s_{ii}) by $s_{pp}^{(qi)}, s_{qq}^{(pi)}, s_{ii}^{(qp)}$. We can then apply the same argument as before as

$$\mathbb{E}[s_{pp}^{(qi)} s_{qq}^{(pi)} K_{pq}^{(pqki)} (\frac{1}{n} \alpha_i^* s^{(qpi)} \alpha_i - \frac{1}{n} \text{Tr} s^{(qpi)})] = 0$$

to conclude that

$$\mathbb{E}[(s_{ii} - s^1(z)) s_{pq}] = O(\frac{v_0^{-20}}{n\sqrt{n}}). \quad (23)$$

- *Estimates of T_1^n and T_5^n .* We first prove that for $v_0 \geq n^{-1/20}$, uniformly on \mathbb{C}_{v_0} , we have

$$\frac{1}{n}T_1^n = \frac{1}{n} \sum_{i,j} \text{Var}(X_i s_{ij} X_j) \rightarrow (\mu - 2)(s^1(z))^2 + \partial_z s^1(z). \quad (24)$$

In fact, let us write the following decomposition:

$$\frac{1}{n}T_1^n = \frac{1}{n} \sum_i \text{Var}(X_i^2 s_{ii}) + \frac{1}{n} \sum_{i \neq j} \text{Var}(X_i X_j s_{ij}) = \frac{1}{n}T_{11}^n + \frac{1}{n}T_{12}^n.$$

To estimate the first term, observe that

$$|\mathbb{E}[s_{ii}^2 - (s^1(z))^2]| = |\mathbb{E}[(s_{ii} - s^1(z))(s_{ii} + s^1(z))]| \leq \frac{2}{v_0} \frac{1}{\sqrt{v_0^4 6n}}$$

by Cauchy-Schwartz inequality and Theorem 9. Hence, $\mathbb{E}[s_{ii}^2] \rightarrow (s^1(z))^2$ for $v_0 \geq n^{-1/19}$. Similarly, we can show that $\mathbb{E}[s_{ii}^2] \rightarrow (s^1(z))^2$. Therefore we deduce that, since $\mu = \mathbb{E}[X_i^4]$, uniformly on $v_0 \geq n^{-1/19}$, we have

$$\frac{1}{n}T_{11}^n = \frac{1}{n} \sum (\mathbb{E}[X_i^4] \mathbb{E}[s_{ii}^2] - \mathbb{E}[s_{ii}^2]^2) \simeq (\mu - 1)(s^1(z))^2. \quad (25)$$

Moreover,

$$T_{12}^n = \sum_{i \neq j} \mathbb{E}[X_i^2 X_j^2] \mathbb{E}[s_{ij}^2] = \sum_{i \neq j} \mathbb{E}[s_{ij}^2] + \sum_{i \neq j} \mathbb{E}[s_{ij}^2] \mathbb{E}[(X_i^2 - 1)(X_j^2 - 1)] = T_{121}^n + T_{122}^n$$

where we can estimate the second term T_{122}^n by noticing that $\mathbb{E}[s_{ij}^2]$ does not depend on ij so that we can factorize it and deduce that

$$T_{122}^n = \sum_{i \neq j} \mathbb{E}[s_{ij}^2] \mathbb{E}[(X_i^2 - 1)(X_j^2 - 1)] = \mathbb{E}[s_{m+1, m+2}^2] \left(\mathbb{E}[(\sum_i (X_i^2 - 1))^2] - \mathbb{E}[\sum_i (X_i^2 - 1)^2] \right)$$

By (19), $\mathbb{E}[s_{m+1, m+2}^2]$ is bounded by $v_0^{-6} n^{-1}$ whereas by we can bound the last term by Cn by concentration of measure, Theorem 11, which yields: for all $\ell \in \mathbb{N}$ and $\delta > 0$

$$\mathbb{P}(|\sum_p X_p| \geq \delta \sqrt{N}) \leq e^{-c_0 \delta^2} \quad \mathbb{E}(|\sum_p (X_p^2 - 1)|^p) \leq C_p \sqrt{n}^p \quad (26)$$

This implies that T_{122}^n is bounded by Cv_0^{-6} . For the first term in T_{12}^n we have

$$\begin{aligned} T_{121}^n &= \sum_{i \neq j} \mathbb{E}[s_{ij}^2] = \mathbb{E}[\sum_{i,j} s_{ij}^2 - \sum_i s_{ii}^2] = \mathbb{E}[\text{Tr}(S_k)_1^2 - \sum_i s_{ii}^2] = \mathbb{E}[\text{Tr}(S_k)_1'] - \sum_i \mathbb{E}[s_{ii}^2] \\ &\sim n(\partial_z s^1(z) - (s^1(z))^2) + O(v_0^{-18}). \end{aligned} \quad (27)$$

Therefore, we deduce (24) from (25) and (27). To prove the convergence of T_5^n , notice that

$$\sum_{i,j} \text{Cov}(X_i s_{ij} X_j, X_j s_{ji} X_i) = \sum_{i,j} \text{Var}(X_i s_{ij} X_j),$$

so that by the previous computation

$$\frac{1}{n}T_5^n = \frac{1}{n} \sum_{i,j} \text{Cov}(X_i s_{ij} X_j, X_j s_{ji} X_i) \rightarrow (\mu - 2)(s^1(z))^2 + \partial_z s^1(z).$$

- *Estimate of T_2^n* .

For the second term, we have by using that $\mathbb{E}[s_{ii}s_{pp}]$ is independent of $i \neq p$.

$$\begin{aligned}
T_2^n &= \sum_{i,p} \text{Cov}(s_{ii}X_i^2, s_{pp}X_p^2) \\
&= \sum_{i,p} (\mathbb{E}[s_{ii}s_{pp}]\mathbb{E}[X_i^2X_p^2] - \mathbb{E}[s_{ii}]\mathbb{E}[s_{pp}]) \\
&= \sum_{i,p} (\mathbb{E}[s_{ii}s_{pp}] - \mathbb{E}[s_{ii}]\mathbb{E}[s_{pp}]) + \sum_{i,p} (\mathbb{E}[X_i^2X_p^2] - 1)\mathbb{E}[s_{ii}s_{pp}] \\
&= \text{Var}(\text{Tr}(S_1)) + \text{Var}\left(\sum_{i=1}^n X_i^2\right)(s^1(z))^2 + o(n).
\end{aligned}$$

Let f be the function $f : x \mapsto (x - z)^{-1}$. f is a Lipschitz function, whose Lipschitz constant is bounded by v_0^{-2} . By Corollary 13 (see the appendix 4.1), we can neglect the variance of $\text{Tr}(S_1)$. Since $\lim \frac{1}{n} \text{Var}(\sum_{i=1}^n X_i^2) = \kappa$, we now have

$$\frac{1}{n} \sum_{i,p} \text{Cov}(s_{ii}X_i^2, s_{pp}X_p^2) \rightarrow \kappa(s^1(z))^2.$$

- *Estimate of T_3^n* Now, we will prove that the other terms can all be neglected and first estimate T_3^n . We start by the following expansion:

$$\begin{aligned}
\sum_i \sum_{p \neq q} \text{Cov}(X_i^2 s_{ii}, X_p s_{pq} X_q) &= \sum_i \sum_{p \neq q} \mathbb{E}[X_i^2 X_p X_q] \mathbb{E}[s_{ii} s_{pq}] \\
&= \sum_i \sum_{p \neq q} \mathbb{E}[(X_i^2 - 1) X_p X_q] \mathbb{E}[s_{ii} s_{pq}]
\end{aligned}$$

We therefore have, by symmetry of $\mathbb{E}[s_{ii}s_{pq}]$, that

$$\begin{aligned}
T_3^n &= \sum_i \sum_{p \neq q} \mathbb{E}[(X_i^2 - 1) X_p X_q] \mathbb{E}[s_{ii} s_{pq}] \\
&= \mathbb{E}[s_{11} s_{23}] \mathbb{E}\left[\left(\sum_i (X_i^2 - 1)\right) \left(\sum_{p \neq q} X_p X_q\right)\right] \\
&\quad + \sum_{(pq)=(12),(21)} (\mathbb{E}[s_{11} s_{pq}] - \mathbb{E}[s_{11} s_{23}]) (\mathbb{E}[\sum_i (X_i^2 - 1) X_i \sum_p X_p] - \mathbb{E}[\sum_i (X_i^2 - 1) X_i^2])
\end{aligned} \tag{28}$$

The last expectations are at most of order $n\sqrt{n}$ by (26) whereas (23) shows that the expectations over the s_{ij} are at most of order $v_0^{-12}/n\sqrt{n}$. Therefore, T_3^n is bounded by $O(v_0^{-14})$.

- *Estimate of T_4^n and T_6^n*

For $i \neq j \neq p \neq q$, since the indices are bigger than m (it is the matrix $(S_k)_1$ which is concerned), we have as for the estimation of T_3^n

$$\mathbb{E}[s_{ij}s_{pq}] = O\left(\frac{1}{v_0^{14}n\sqrt{n}}\right).$$

Moreover, this term does not depend on the choices of $i \neq j \neq p \neq q$ and we also have by concentration of measure

$$\sum_{i \neq j \neq p \neq q} \mathbb{E}[X_i X_j X_p X_q] = O(n^2).$$

Therefore,

$$\frac{T_6^n}{n} = \frac{1}{n} \sum_{i \neq j \neq p \neq q} \text{Cov}(X_i s_{ij} X_j, X_p s_{pq} X_q) = O\left(\frac{1}{v_0^{14} \sqrt{n}}\right).$$

Finally, using the same reasoning, we show that

$$\frac{T_4^n}{n} = \frac{1}{n} \sum_i \sum_{j \neq p} \text{Cov}(X_i s_{ij} X_j, X_p s_{pi} X_i) = O\left(\frac{1}{v_0^{18} \sqrt{n}}\right).$$

Indeed, if $i \neq j \neq p$, we know that $\mathbb{E}[s_{ij} s_{pi}] = O\left(\frac{v_0^{-18}}{n \sqrt{n}}\right)$. And, if $i = j$ or $i = p$, we have $\mathbb{E}[s_{ii} s_{pi}] = O\left(\frac{v_0^{-9}}{n}\right)$. In both cases, $\mathbb{E}[s_{ij} s_{pi}] = O\left(\frac{v_0^{-18}}{n}\right)$, which ends the proof.

To conclude, for $k \leq m$, we have proved that uniformly on $v_0 \geq n^{-1/36}$,

$$\lim_{(n,m) \rightarrow \infty, m/n \rightarrow c} m \mathbb{E}[\epsilon_k^2] = 2c(\mu - 2 + \frac{\kappa}{2})(s^1(z))^2 + 2c \partial_z s^1(z).$$

This aim plies that

$$\mathbb{E}[\delta_2(z)] = -\frac{c}{m}(s^2(z))^2 \frac{\partial_z s^2(z)}{s^2(z)} + \frac{c}{m}(s^2(z))^3 \left[2(\mu - 2 + \frac{\kappa}{2})(s^1(z))^2 + 2\partial_z s^1(z) \right].$$

For $k > m$, using the same method, we have uniformly on $v_0 \geq n^{-1/36}$,

$$n \mathbb{E}[\epsilon_k] \rightarrow \frac{\partial_z s^1(z)}{s^1(z)}.$$

This time, for $\mathbb{E}[\epsilon_k]$, $k > m$, the computations are almost the same as previously, except for the fact that the vectors are now independent. Therefore, we have $\text{Var}(\sum_i X_i^2) = m(\mu - 1)$. Thus, for $k > m$, we have the uniform convergence for $v_0 \geq n^{-1/36}$;

$$\lim_{(n,m) \rightarrow \infty, m/n \rightarrow c} n \mathbb{E}[\epsilon_k^2] = c \left((4\mu - 5)(s^2(z))^2 + 2\partial_z s^2(z) \right)$$

.

Summing these estimates, we deduce that uniformly on $v_0 \geq n^{-1/36}$,

$$\mathbb{E}[\delta_1(z)] \sim -\frac{1}{n}(s^1(z))^2 \frac{\partial_z s^1(z)}{s^1(z)} + \frac{c}{n}(s^1(z))^3 \left[(4\mu - 5)(s^2(z))^2 + 2\partial_z s^2(z) \right].$$

Noticing that $m = nc + O(1)$, we have

$$n(\mathbb{E}[s_{n,m}^1(z)] - s^1(z)) + m(\mathbb{E}[s_{n,m}^2(z)] - s^2(z)) = n[c_{n,m} - c]A(z) + n\mathbb{E}[\delta_1(z)]B(z) + m\mathbb{E}[\delta_2(z)]C(z) + O(n^{-1})$$

where

$$\begin{aligned} A(z) &= -\frac{(z^2 + c - 1)}{2zc} + \frac{1}{2\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[\frac{z^2 - c + 1}{z^2} - \frac{z^2 + c - 1}{cz^2}(z^2 - 1) + 2 \right] \\ B(z) &= \frac{1}{2} + \frac{c - 1}{2z^2} + \frac{1}{2\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[z - \frac{(c - 1)^2}{z^3} \right] \\ C(z) &= \frac{1}{2} - \frac{c - 1}{2z^2} + \frac{1}{2\sqrt{\frac{(z^2 - c + 1)^2}{z^2} - 4}} \left[z - \frac{(c - 1)^2}{z^3} \right] \end{aligned}$$

But, using the expressions of $s^1(z)$ and $s^2(z)$, we can see that

$$A(z) = s^2(z), B(z) = 1 + c\partial_z s^2(z), C(z) = 1 + \partial_z s^1(z).$$

Therefore, if we define $M(z)$ by

$$\begin{aligned} M(z) := & \sigma s^2(z) + (1 + c\partial_z s^2(z))(s^1(z))^3 \left[-\frac{\partial_z s^1(z)}{(s^1(z))^2} + c \left[(4\mu - 5)(s^2(z))^2 + 2\partial_z s^2(z) \right] \right] \\ & + (1 + \partial_z s^1(z))c(s^2(z))^3 \left[-\frac{\partial_z s^2(z)}{(s^2(z))^2} + \left[2(\mu - 2 + \frac{\kappa}{2})(s^1(z))^2 + 2\partial_z s^1(z) \right] \right] \end{aligned}$$

we have proved that uniformly on $v_0 \geq n^{-1/36}$,

$$M_{n,m}(z) \rightarrow M(z).$$

□

4 Appendix

4.1 Concentration of measure

Theorem 11. *Let X be a random vector whose distribution satisfies Hypothesis 1 and (2). For all symmetric matrix A , for all integer p , there exist $C_p < \infty$ such that*

$$\mathbb{E}[|X^*AX - \text{Tr}(A)|^p] \leq C_p \|A\|^p \sqrt{n}^p \quad \|A\| = \sup_{\|y\|=1} \|Ay\|, \quad \|\cdot\| \text{ euclidean norm.}$$

Proof. We decompose the covariance as :

$$\mathbb{E}[|X^*AX - \text{Tr}(A)|^p] \leq 2^{p-1} (\mathbb{E}[|\sum_{i=1}^N (X_i^2 - \mathbb{E}[X_i^2])a_{ii}|^p] + \mathbb{E}[|\sum_{i \neq j} X_i X_j a_{ij}|^p]). \quad (29)$$

We can now find an upper bound for each term. For this, we will use a remarkable property of V . Since $\text{Hess } V \geq \frac{1}{C} I_n$, the logarithmic Sobolev inequality holds for the measure dX with the constant C (see e.g. [1, Lemma 2.3.2 and 2.3.3]). Consequently, for all Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if we write $|f|_{\mathcal{L}}$ its Lipschitz constant, we have

$$\mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq t) \leq 2e^{\frac{-t^2}{2C|f|_{\mathcal{L}}^2}}.$$

Let us focus on the first term on the right hand side of (29).

First, let us suppose that for all i , $a_{ii} \geq 0$. Let $f : x \mapsto \sqrt{\sum a_{ii}x_i^2}$, we will show that this is a Lipschitz function whose constant is bounded by $\|A\|_{\infty}^{\frac{1}{2}}$. Indeed, $\frac{\partial f}{\partial x_i} = a_{ii} \frac{x_i}{f(x)}$. Therefore, since $a_{ii} \geq 0$, we have $\|\nabla f\|^2 \leq \max |a_{ii}| \leq \|A\|_{\infty}$. Taylor's inequality states that $|f|_{\mathcal{L}} \leq \|A\|_{\infty}^{\frac{1}{2}}$. For all $p \in \mathbb{N}$, we now have :

$$\begin{aligned} \mathbb{E}[|\sqrt{\sum a_{ii}X_i^2} - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]|^p] &= \int_0^\infty t^{p-1} \mathbb{P}(|\sqrt{\sum a_{ii}X_i^2} - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]| \geq t) dt \\ &\leq 2 \int_0^\infty t^{p-1} e^{\frac{-t^2}{2\|A\|_{\infty}^2 C}} dt \leq C_p \|A\|_{\infty}^{\frac{p}{2}}. \end{aligned}$$

Thus, we have :

$$C_1 \leq 2^{p-1}(\mathbb{E}[|\sum a_{ii}X_i^2 - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]^2|^p] + \mathbb{E}[|\mathbb{E}[\sum a_{ii}X_i^2] - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]^2|^p]),$$

By the concentration inequality, we know that the second term is bounded regardless of n . Then, for the first term, Cauchy-Schwartz inequality gives

$$\begin{aligned} \mathbb{E}[|\sum a_{ii}X_i^2 - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]^2|^p] &= \mathbb{E}[|\sqrt{\sum a_{ii}X_i^2} - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]|^p |\sqrt{\sum a_{ii}X_i^2} + \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]|^p], \\ &\leq \mathbb{E}[|\sqrt{\sum a_{ii}X_i^2} - \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]|^{2p}]^{\frac{1}{2}} \mathbb{E}[|\sqrt{\sum a_{ii}X_i^2} + \mathbb{E}[\sqrt{\sum a_{ii}X_i^2}]|^{2p}]^{\frac{1}{2}}. \end{aligned}$$

Using again the concentration inequality, we show that the first term of the product can be bounded by the product of $\|A\|_\infty^p$ and a constant depending only on p . Assembling all these terms alongside with noticing that $\|A\|_\infty \leq \|A\|$, we have proved that there exist C_p independent of n such that :

$$\mathbb{E}[|\sum_{i=1}^n (X_i^2 - \mathbb{E}[X_i^2])a_{ii}|^p] \leq C_p \|A\|^p \sqrt{n}^p.$$

Now, for the general case, let us write A as a sum of a positive and a negative matrix $A = A^+ - A^-$. Therefore,

$$\begin{aligned} \mathbb{E}[|\sum_{i=1}^N (X_i^2 - \mathbb{E}[X_i^2])a_{ii}|^p] &= \mathbb{E}[|\sum_{i=1}^N (X_i^2 - \mathbb{E}[X_i^2])a_{ii}^+ - a_{ii}^-|^p] \\ &\leq 2^{p-1}(\mathbb{E}[|\sum_{i=1}^N (X_i^2 - \mathbb{E}[X_i^2])a_{ii}^+|^p] + \mathbb{E}[|\sum_{i=1}^N (X_i^2 - \mathbb{E}[X_i^2])a_{ii}^-|^p]). \end{aligned}$$

Since the a_{ii}^+ and the a_{ii}^- are all non negative, the previous proof still holds and therefore, we have

$$\mathbb{E}[|\sum_{i=1}^n (X_i^2 - \mathbb{E}[X_i^2])a_{ii}|^p] \leq C_p \|A\|^p \sqrt{n}^p.$$

On the other hand, if we denote by Z_n the cardinal number of partitions of $\llbracket 1, n \rrbracket$ into two distincts sets I and J , we have

$$\mathbb{E}[|\sum_{i \neq j} X_i X_j a_{ij}|^p] \leq \frac{1}{Z_n} \sum_{I \cap J = \emptyset} \mathbb{E}[|\sum_{(i,j) \in (I \times J)} X_i X_j a_{ij}|^p].$$

We next expand the right hand side as

$$\mathbb{E}[|\sum_{(i,j) \in (I \times J)} X_i X_j a_{ij}|^p] = \sum_{i_1, \dots, i_p \in I} \sum_{j_1, \dots, j_p \in I} \prod_{k=1}^p a_{i_k, j_k} \mathbb{E}[X_{i_1} \cdots X_{i_p} X_{j_1} \cdots X_{j_p}]$$

In the above right hand side, assume that there are K indices s_1, \dots, s_K with multiplicity one. Let ℓ_1, \dots, ℓ_r be the others. We then can apply the symmetry of the law of the X_i 's to see that for some $n_1, \dots, n_r \geq 2$,

$$\mathbb{E}[X_{i_1} \cdots X_{i_p} X_{j_1} \cdots X_{j_p}] = \mathbb{E}[X_{\ell_1}^{n_1} \cdots X_{\ell_r}^{n_r} \prod_{k=1}^K X_{s_k}] = \mathbb{E}[X_1^{n_1} \cdots X_{\ell_r}^{n_r} \prod_{k=1}^K \frac{1}{N_K} \sum_{s \in I_k} X_{s_k}]$$

where $I_k = [r+1+(k-1)N_K, r+kN_K]$ and $KN_K + \sum n_i = n$. Now, by concentration inequality applied to $\sum_{s \in I_k} X_{s_k}$ we deduce that there exists a finite constant C_p such that

$$|\mathbb{E}[X_{i_1} \cdots X_{i_p} X_{j_1} \cdots X_{j_p}]| \leq \sqrt{n}^{-K}.$$

Using that a_{ij} is bounded by $\|A\|$, we conclude that

$$\mathbb{E}[\left|\sum_{(i,j) \in (I \times J)} X_i X_j a_{ij}\right|^p] \leq C_p \sum_{K=0}^p \|A\|^p n^{K/2} n^{\frac{p-K}{2}} \leq C_p \|A\|^p n^p$$

which completes the proof. \square

Remark 12. Let $1 \leq k \leq n$. The logarithmic Sobolev inequality holds for each $X_i(k), i \in \llbracket 1, m \rrbracket$ with the constant C . By independence of the $X_i(k), i \in \llbracket 1, m \rrbracket$, it also holds for the vector $(X_1(k), \dots, X_m(k))$ with the same constant. Therefore, the concentration inequality holds the same way for the vector $(X_1(k), \dots, X_m(k))$ for all $k, 1 \leq k \leq n$.

Moreover, the spectral measure satisfies concentration inequalities [1, Theorem 2.3.5]:

Corollary 13. Under Hypothesis 1, for all Lipschitz function f on \mathbb{R} , for all $p \in \mathbb{N}$, for all $(n, m) \in \mathbb{N}^2$, there exist C_p independent of n and m such that

$$\mathbb{E}[|\text{Tr}(f(W_{n,m})) - \mathbb{E}[\text{Tr}(f(W_{n,m}))]|^p] \leq C_p |f|_{\mathcal{L}}^p$$

.

The final result we will need is the concentration of the extremal eigenvalues, see [1, Corollary A.6]

Theorem 14. Let λ_i be the ordered eigenvalues of $W_{n,m}$. Then there exists a positive constant c_0 and a finite constant C such that for all N

$$\mathbb{P}(|\lambda_i - \mathbb{E}[\lambda_i]| \geq \delta) \leq C \exp\{-c_0 N \delta^2\}$$

4.2 Boundedness of the spectrum of $W_{n,m}$

We will now show that, asymptotically, the spectrum of $W_{n,m}$ is bounded. For this, we will compare the law of the matrix with the one of a Wigner matrix whose entries are all i.i.d centered Gaussian random variables.

Indeed,

Lemma 15. Under Hypothesis 1, there exist $\alpha > 0$ and $C < \infty$ such that, for all integer n , we have

$$\mathbb{P}(\lambda_{\max}(W_{n,m}) > C) \leq e^{-\alpha C n}.$$

Proof. Since $W_{n,m}$ is symmetric, the law of $W_{n,m}$ is the law of $Y_{n,m}$ which we can write as :

$$dY_{n,m} = \frac{1}{Z_n} \prod_{j=1}^m e^{-V(\sqrt{n}(Y_{1j}, \dots, Y_{nj}))} \prod_i^n \prod_j^m dY_{i,j}.$$

By (2) this law has a strictly log-concave density and therefore, if we denote by γ the Gaussian law

$$d\gamma = \frac{1}{Z_C} \prod_i^n \prod_j^m e^{-\frac{n}{2C} Y_{ij}^2} dY_{i,j},$$

we can apply Brascamp-Lieb inequality ([10, Thm 6.17]) which implies that for all convex function g , we have :

$$\int g(x) \frac{f(x) d\gamma(x)}{\int f d\gamma} \leq \int g(x) d\gamma(x).$$

Applying this inequality with $g(x) = e^{ns\lambda_{\max}(W_{n,m})}$ for $s > 0$ we deduce that

$$\int e^{sn\lambda_{\max}(W_{n,m})} dY_{n,m} \leq \int e^{sn\lambda_{\max}(W_{n,m})} d\gamma.$$

The right hand side is bounded by C^n for some finite constant C , see e.g. [1, Section 2.6.2]. Tchebychev's inequality completes the proof. \square

4.3 An example for the function V

We will show in this part that there exists V such that the hypotheses made in the introduction can be verified without jeopardizing the dependance of the random variables, i.e. there exists V such that κ is different from $\mu - 1$ (which is the result expected for the case where the random variables are independent).

Let V be given by

$$V : (X_1, \dots, X_n) \mapsto \frac{a}{n} \left(\sum_{i=1}^n (X_i^2 - m_{b,c}) \right)^2 + b \sum_{i=1}^n X_i^2 + c \sum_{i=1}^n X_i^4.$$

Therefore, the law of the random vector X can be written as

$$dX = \frac{1}{Z_{a,b,c}^n} e^{\left(-\frac{a}{n} \left(\sum_{i=1}^n (X_i^2 - m_{b,c})\right)^2 - b \sum_{i=1}^n X_i^2 - c \sum_{i=1}^n X_i^4\right)} \prod_{i=1}^n dX_i$$

where, if we denote by $\mathbb{E}_{a,b,c}[\cdot]$ the expectation under this measure, $m_{b,c} = \mathbb{E}_{0,b,c}[X_i^2]$. This law has obviously a strictly log-concave density when a, b, c are positive, and it is symmetric. Hence, Hypothesis 1 is fulfilled.

First, let us note that for all $a \geq 0$, we have

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow m_{b,c}.$$

Now, let us estimate $Z_{a,b,c}^n$. For this, we will introduce a Gaussian distribution $\mathcal{N}(0, 1)$ as it follows :

$$Z_{a,b,c}^n = Z_{0,b,c}^n \frac{1}{\sqrt{2\pi}} \int e^{-\frac{g^2}{2}} \left(\mathbb{E}_{0,b,c} \left[e^{\left(\frac{i\sqrt{a}g}{\sqrt{n}} (X_i^2 - m_{b,c})\right)} \right] \right)^n dg.$$

Thus, we have

$$Z_{a,b,c}^n \simeq Z_{0,b,c}^n \frac{1}{\sqrt{2\pi}} \int e^{-\frac{g^2}{2}} e^{-\frac{a^2}{2} g^2 \sigma_{b,c} + O(1/\sqrt{n})} dg,$$

where

$$\sigma_{b,c} = \mathbb{E}_{0,b,c} \left[(X_i^2 - m_{b,c})^2 \right].$$

Therefore,

$$\log(Z_{a,b,c}^n) = \log(Z_{0,b,c}^n) - \frac{1}{2} \log(1 + a^2 \sigma_{b,c}) + O(1).$$

On the other side, we have

$$\begin{aligned}
\mathbb{E}_{a,b,c}[X_i X_j] &= 1_{i=j} \mathbb{E}_{a,b,c}[X_i^2] \\
\mathbb{E}_{a,b,c}[X_i^2] &= -\frac{1}{n} \partial_b \log(Z_{a,b,c}^n) = \mathbb{E}_{0,b,c}[X_i^2] + O\left(\frac{1}{n}\right) \\
\mathbb{E}_{a,b,c}[X_i^4] &= -\frac{1}{n} \partial_c \log(Z_{a,b,c}^n) = \mathbb{E}_{0,b,c}[X_i^2] + O\left(\frac{1}{n}\right) \\
\frac{1}{n} \mathbb{E}_{a,b,c}[(\sum (X_i^2 - \mathbb{E}_{a,b,c}[X_i^2]))^2] &= \frac{1}{n} \mathbb{E}_{a,b,c}[(\sum (X_i^2 - m_{b,c}))^2] - n(m_{b,c} - \mathbb{E}_{a,b,c}[X_i^2])^2 \\
&\sim -\frac{1}{n} \partial_a \log(Z_{a,b,c}^n) + O\left(\frac{1}{n}\right)
\end{aligned}$$

Summarizing what we have done, we can set the value of $\kappa = \frac{1}{n} \text{Var}(\sum X_i^2)$ with the parameter a , regardless of the value of $\mu = \mathbb{E}[X_i^4]$. It is not hard to see that the symmetry condition of Hypothesis ?? is fulfilled.

4.4 Linear Algebra

In this section we remind a few classical linear algebra identities.

Lemma 16. *Let A be a square invertible matrix, and if A is a block matrix, its determinant can be computed using the following formula :*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$

Lemma 17. *For $n \times n$ Hermitian nonsingular matrix A , define A_k , which is a matrix of size $(n-1)$, to be the matrix resulting from deleting the k -th row and column of A . If both A and A_k are invertible, denoting $A^{-1} = (a^{ij})$ and α_k the vector obtained from the k -th column of A by deleting the k -th entry, we have*

$$a^{kk} = \frac{1}{a_{kk} - \alpha_k^* A_k^{-1} \alpha_k}.$$

More precisely, if for all k , A_k is invertible, we have

$$\text{Tr}(A^{-1}) = \sum_{i=1}^n \frac{1}{a_{ii} - \alpha_i^* A_i^{-1} \alpha_i}.$$

Lemma 18. *If the matrix A and A_k are both nonsingular and hermitian, we have*

$$\text{Tr}(A^{-1}) - \text{Tr}(A_k^{-1}) = \frac{1 + \alpha_k^* A_k^{-2} \alpha_k}{a_{kk} - \alpha_k^* A_k^{-1} \alpha_k}.$$

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